Estimation of Skill Distribution from a Tournament

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England cricket team

Lord's the perfect venue for greatest one-day game ever played

Andy Bull *at Lord's*

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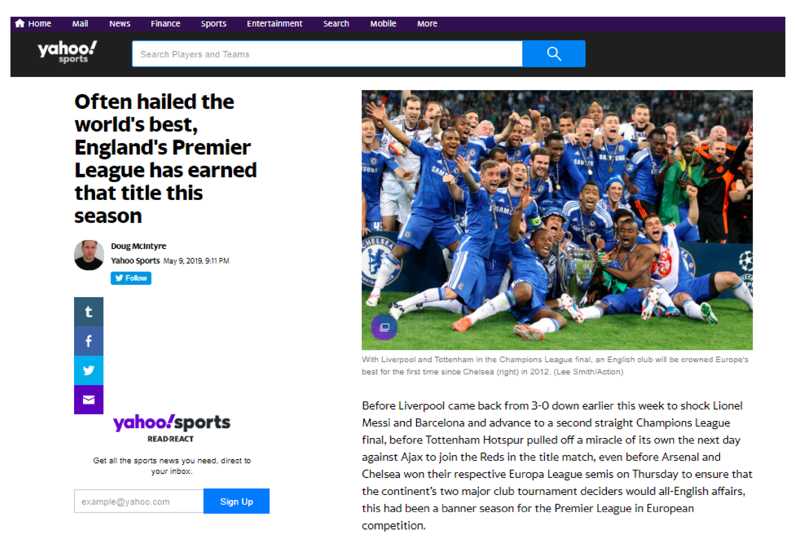
Where better for England's transgenerational trauma to end than the home of cricket, pretty as a picture in the evening sun

The sun was out when England won the World Cup. It had emerged, at last, from behind the freckles of white cloud and was shining low through the bright blue sky over the Grand Stand. The shadows stretched all the way to the wicket, the flags licked in the evening breeze, the pavilion glowed soft terracotta. The old place looked pretty as a picture, exactly how we see it in winter when thinking back on the games we saw, and the games we played, in the long, warm days of summer. For the 11 men in this England team, for the thousands in the ground, for the millions watching on TV, that is how they will always see it in their minds, years from now, when they think back on this match: the greatest World Cup final, heck



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Often hailed the world's best, England's Premier League has earned that title this season

Doug McIntyre

Yahoo Sports May 9, 2019, 9:11 PM

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With Liverpool and Tottenham in the Champions League final, an English club will be crowned Europe's best for the first time since Chelsea (right) in 2012. (Lee Smith/Action)

Before Liverpool came back from 3-0 down earlier this week to shock Lionel Messi and Barcelona and advance to a second straight Champions League final, before Tottenham Hotspur pulled off a miracle of its own the next day against Ajax to join the Reds in the title match, even before Arsenal and Chelsea won their respective Europa League semis on Thursday to ensure that the continent's two major club tournament deciders would all-English affairs, this had been a banner season for the Premier League in European competition.

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- Can we formalize such claims about sports using statistical analysis?
- Can we measure the overall level of skill in a game based on win-loss data from tournaments?

Formal Setup and Goal

- Unknown **PDF of skill levels** P_α on interval $[\delta, 1]$ for $\delta > 0$, which is bounded and belongs to η-Hölder class.
- Teams $\{1, \ldots, n\}$ play tournament with unknown i.i.d. skill levels $\alpha_1, \ldots, \alpha_n \sim P_\alpha$.
- For $i \neq j$, with probability $p \in (0, 1]$, observe k independent pairwise games where $Z_m(i, j) = \mathbb{I}\{j \text{ beats } i \text{ in } m\text{th game}\}$.
- **Bradley-Terry-Luce (BTL)** or **multinomial logit** model [1]:

$$\mathbb{P}(Z_m(i,j) = 1 \,|\, \alpha_1, \ldots, \alpha_n) = \frac{\alpha_j}{\alpha_i + \alpha_j}.$$

- **Goal:** Learn P_α from observation matrix $Z \in [0, 1]^{n \times n}$ with

$$Z(i,j) = \begin{cases} \dfrac{1}{k} \displaystyle\sum_{m=1}^{k} Z_m(i,j)\,, & \text{if games observed between } i \neq j\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

- **Overall skill score:** Negative differential entropy of P_α

$$-h(P_\alpha) = \int_\delta^1 P_\alpha(x) \log(P_\alpha(x))\, \mathrm{d}x$$

measures the variation of skill levels of teams in a tournament.

- **Intuition:** Concentrated P_α has high score and outcomes of games are unpredictable; Balanced P_α has low score and there is more variation of skill levels.

Estimation Algorithm

Estimation of P_α from Z

Input: Observation matrix Z

Output: Estimator $\widehat{\mathcal{P}}^*$ of P_α

Step 1: **Skill parameter estimation using rank centrality algorithm** [2]

1. Construct stochastic matrix $S \in \mathbb{R}^{n \times n}$ with $S(i,j) = \frac{Z(i,j)}{2np}$ for $i \neq j$, whose rows sum to 1
2. Compute leading left eigenvector $\hat{\pi}_*$ of S such that $\hat{\pi}_* = \hat{\pi}_* S$
3. Compute skill level estimates $\hat{\alpha}_i = \frac{\hat{\pi}_*(i)}{\|\hat{\pi}_*\|_\infty}$ for $i = 1, \ldots, n$

Step 2: **Kernel density estimation using Parzen-Rosenblatt method**

4. Compute bandwidth $h = \Theta\big(\log(n)^{\frac{1}{2\eta+2}} n^{-\frac{1}{2\eta+2}}\big)$
5. Construct $\widehat{\mathcal{P}}^*$ using appropriate, fixed kernel $K : [-1, 1] \to \mathbb{R}$

$$\widehat{\mathcal{P}}^*(x) \triangleq \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{\hat{\alpha}_i - x}{h}\right)$$

6. **Return** $\widehat{\mathcal{P}}^*$

Theoretical Results

Theorem (Mean Squared Error Upper Bound)

If $p = \Omega(\log(n)/(\delta^5 n))$ and $\lim_{n \to \infty} \delta^{-1}(npk)^{-1/2} \log(n)^{1/2} = 0$, then for all P_α,

$$\mathbb{E}\left[\int_{\mathbb{R}} \left(\widehat{\mathcal{P}}^*(x) - P_\alpha(x)\right)^2 \mathrm{d}x\right] = O\left(\max\left\{\left(\frac{\log(n)}{\delta^2 pkn}\right)^{\frac{\eta}{\eta+1}}, \left(\frac{\log(n)}{n}\right)^{\frac{\eta}{\eta+1}}\right\}\right).$$

- **Summary of all minimax estimation results:**

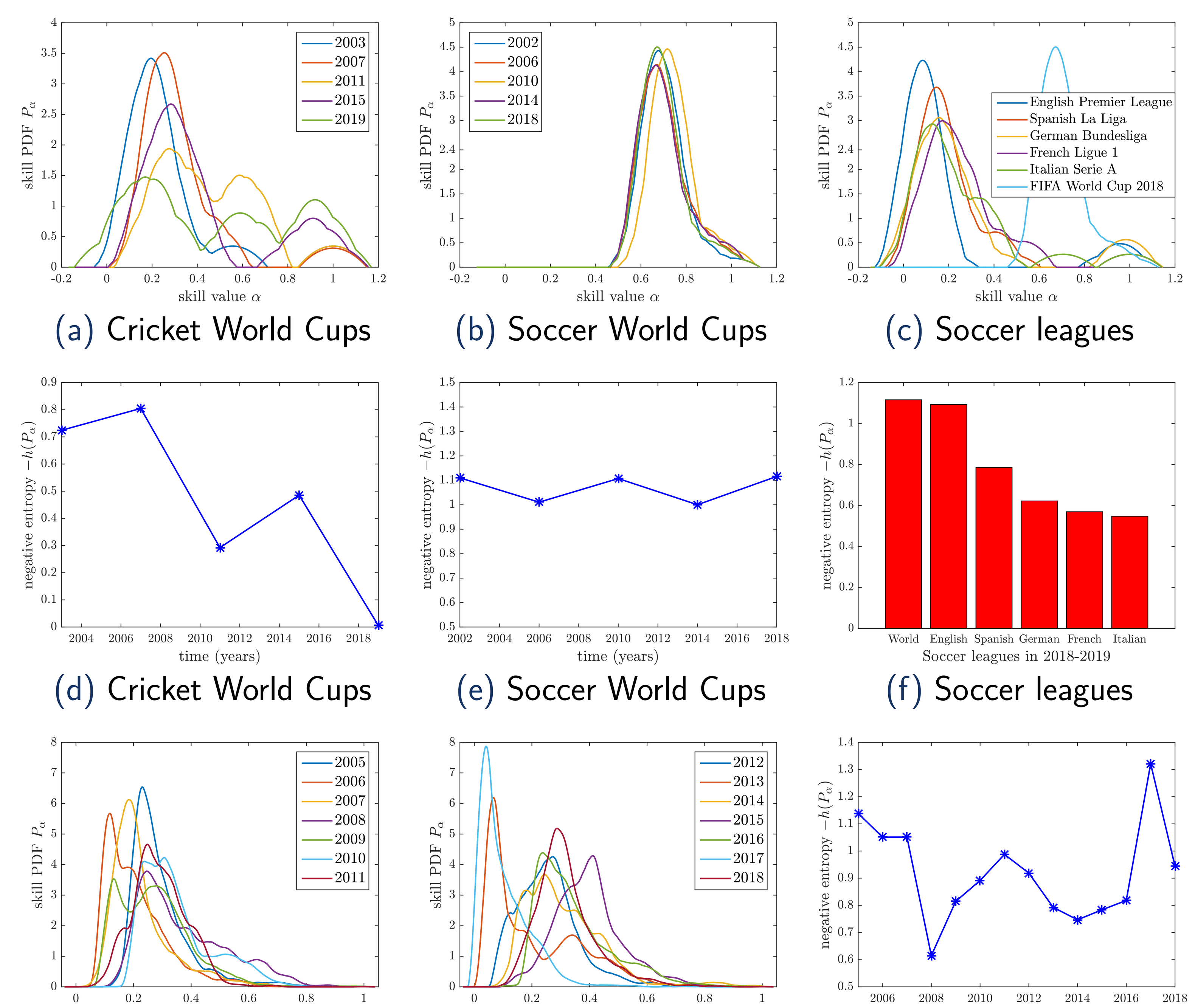
Estimation prob.	Loss func.	Upp. bound	Low. bound
Smooth skill PDF	MSE	$\tilde{O}(n^{-1+\varepsilon})$	$\Omega(n^{-1})$
BTL skill levels	ℓ^∞-norm	$\tilde{O}(n^{-1/2})$	$\tilde{\Omega}(n^{-1/2})$
BTL skill levels	ℓ^1-norm	$O(n^{-1/2})$	$\tilde{\Omega}(n^{-1/2})$

- Note: Our results are in red; other results are known in the literature. Notation $\tilde{O}$ and $\tilde{\Omega}$ hide polylog(n) terms, and $\varepsilon > 0$ is any arbitrarily small constant.

Experiments

- **Cricket world cups:** Skill scores of cricket world cup tournaments are decreasing over time.
- **Soccer world cups:** Soccer world cups have remained quite unpredictable over the years.
- **Soccer leagues in 2018-2019:** Recover ranking of soccer leagues that is consistent with fan experience.
- **US mutual funds:** Skill score is minimum during the *Great Recession* of 2008.

Figure: Plots (a), (b), (c), (g), and (h) illustrate *estimated skill PDFs*, and plots (d), (e), (f), and (i) depict corresponding *negative differential entropies*.



(a) Cricket World Cups (b) Soccer World Cups (c) Soccer leagues

(d) Cricket World Cups (e) Soccer World Cups (f) Soccer leagues

(g) US mutual funds (h) US mutual funds (i) US mutual funds

References

[1] R. A. Bradley and M. E. Terry, "Rank analysis of incomplete block designs. I. The method of paired comparisons," *Biometrika*, vol. 39, no. 3/4, pp. 324–345, December 1952.

[2] S. Negahban, S. Oh, and D. Shah, "Rank centrality: Ranking from pairwise comparisons," *Operations Research, INFORMS*, vol. 65, no. 1, pp. 266–287, January-February 2017.